ZAPOREDJA: NARAŠČANJE IN PADANJE

NARAŠČA: $a_{n+1} - a_n > 0$
PADA: $a_{n+1} - a_n < 0$

1. Pokaži, da je zaporedje $a_n = 4n - 5$ naraščajoče.

Izračunaš $a_{n+1} - a_n$

namesto n vstaviš $n+1$

$a_n = 4n - 5$
$a_{n+1} = 4(n+1) - 5$

obvezno v oklepaj

$a_{n+1} - a_n = 4(n+1) - 5 - (4n-5) = 4n + 4 - 5 - 4n + 5 = 4 > 0$ ✓

NARAŠČAJOČE
ker je $a_{n+1} - a_n > 0$

2. Pokaži, da je zaporedje $a_n = n^2 + n$ naraščajoče.

$a_n = n^2 + n$
$a_{n+1} = (n+1)^2 + n + 1$

vstaviš $n+1$ namesto vsakega n

$(a+b)^2 = a^2 + 2ab + b^2$

$a_{n+1} - a_n = (n+1)^2 + n + 1 - (n^2 + n) =$

$= \cancel{n^2} + 2n + 1 + \cancel{n} + 1 - \cancel{n^2} - \cancel{n} = 2n + 2 > 0$ ✓

NARAŠČAJOČE

n je naravno število $(1, 2, 3, 4 \ldots)$, torej je $2n+1$ zagotovo večje od 0

$2n + 2 > 0$ za vsak $n \in \mathbb{N}$

3. Pokaži, da je zaporedje $a_n = \frac{2n+4}{3n+5}$ padajoče.

$a_n = \frac{2n+4}{3n+5}$

$a_{n+1} = \frac{2(n+1)+4}{3(n+1)+5} = \frac{2n+2+4}{3n+3+5} = \frac{2n+6}{3n+8}$

namesto n vstaviš n+1

$a_{n+1} - a_n = \frac{2n+6}{3n+8} - \frac{2n+4}{3n+5} =$

daš na skupni imenovalec

$$= \frac{(2n+6)(3n+5)-(2n+4)(3n+8)}{(3n+8)(3n+5)} =$$

OBVEZNO V OKLEPAJ (ker se vsem spremeni predznak)

$$= \frac{6n^2+10n+18n+30-(6n^2+16n+12n+32)}{(3n+8)(3n+5)} =$$

$$= \frac{\cancel{6n^2}+\cancel{10n}+\cancel{18n}+30-\cancel{6n^2}-\cancel{16n}-\cancel{12n}-32}{(3n+8)(3n+1)} = \frac{-2^{\ominus}}{\underset{\oplus}{(3n+8)}\underset{\oplus}{(3n+1)}} < 0$$

zaporedje je padajoče

$-2 < 0$

$3n+8 > 0$

$3n+1 > 0$

ker je $n \in \mathbb{N}$ $(1, 2, 3, 4, \ldots)$

4. Dano je zaporedje s splošnim členom $a_n = 5 \cdot 2^{n-1} + 2$. Je zaporedje naraščajoče ali padajoče?

znamza5si

$a_n = 5 \cdot 2^{n-1} + 2$

$a_{n+1} = 5 \cdot 2^{n+1-1} + 2 = 5 \cdot 2^n + 2$

$a_{n+1} - a_n = 5 \cdot 2^n + 2 - (5 \cdot 2^{n-1} + 2) = 5 \cdot 2^n + 2 - 5 \cdot 2^{n-1} - 2 =$

$= 5 \cdot 2^n - 5 \cdot 2^{n-1} = 5 \cdot 2^{n-1}(2^1 - 1) = 5 \cdot 2^{n-1} > 0 \rightarrow$ naraščajoče

5. Dano je zaporedje s splošnim členom $a_n = 1 + \frac{2}{n}$. Ali je dano zaporedje monotono (naraščajoče / padajoče)?

$a_n = 1 + \frac{2}{n}$

$a_{n+1} = 1 + \frac{2}{n+1}$

$$a_{n+1} - a_n = 1 + \frac{2}{n+1} - \left(1 + \frac{2}{n}\right) = \cancel{1} + \frac{2}{n+1} - \cancel{1} - \frac{2}{n} =$$

$$= \frac{2}{n+1} - \frac{2}{n} = \frac{2n - 2(n+1)}{n(n+1)} = \frac{2n - 2n - 2}{n(n+1)} = \frac{-2}{n(n+1)} < 0$$

$n \in \mathbb{N}$ $(1, 2, 3, 4, \ldots)$, torej sta n in $n+1$ pozitivna

Zaporedje je monotono (padajoče).

6. Dokaži, da je zaporedje s splošnim členom $a_n = \frac{2n-7}{3n+2}$ naraščajoče.

$a_n = \frac{2n-7}{3n+2}$

$$a_{n+1} = \frac{2(n+1)-7}{3(n+1)+2} = \frac{2n+2-7}{3n+3+2} = \frac{2n-5}{3n+5}$$

$$a_{n+1} - a_n = \frac{2n-5}{3n+5} - \frac{2n-7}{3n+2} = \frac{(2n-5)(3n+2) - (2n-7)(3n+5)}{(3n+5)(3n+2)} =$$

$$= \frac{6n^2 + 4n - 15n - 10 - (6n^2 + 10n - 21n - 35)}{(3n+5)(3n+2)} =$$

$$= \frac{\cancel{6n^2} - \cancel{11n} - 10 - \cancel{6n^2} - \cancel{10n} + \cancel{21n} + 35}{(3n+5)(3n+2)} = \frac{25}{(3n+5)(3n+2)} > 0$$

Ker je $n \in \mathbb{N}$

naraščajoče

7. Dano je zaporedje $a_n = \frac{3^n}{2^{n+1}}$. Ali je dano zaporedje monotono?

$$a_n = \frac{3^n}{2^{n+1}}$$

$$a_{n+1} = \frac{3^{n+1}}{2^{n+1+1}} = \frac{3^{n+1}}{2^{n+2}}$$

$$a_{n+1} = \frac{3^{n+1}}{2^{n+2}} - \frac{3^n}{2^{n+1}} = \frac{3^{n+1}}{2 \cdot 2^{n+1}} - \frac{3^n}{2^{n+1}} = \frac{3^{n+1} - 2 \cdot 3^n}{2 \cdot 2^{n+1}} = \frac{3^n (3^1 - 2)}{2 \cdot 2^{n+1}} =$$

$$2^{n+2} = 2^{n+1+1} = 2^{n+1} \cdot 2^1 = 2 \cdot 2^{n+1}$$

$$= \frac{3^n \, (+)}{2 \cdot 2^{n+1} \, (+)} > 0$$

za vsak $n \in \mathbb{N}$ Je monotono.

naraščajoče

če ne vidiš, da je $2^{n+2} = 2 \cdot 2^{n+1}$ lahko daš na skupni imenovalec $2^{n+2} \cdot 2^{n+1}$ (bistvo je, da dokažeš + ali – ne pa da čim lepše zgleda)

$$\frac{3^{n+1} \cdot 2^{n+1} - 3^n \cdot 2^{n+2}}{2^{n+2} \cdot 2^{n+1}} = \frac{3^n \cdot 3^1 \cdot 2^n \cdot 2^1 - 3^n \cdot 2^n \cdot 2^2}{2^{n+2} \cdot 2^{n+1}} =$$

$$3^{n+1} = 3^n \cdot 3^1$$

$$= \frac{3^n \cdot 2^n (3 \cdot 2 - 2^2)}{2^{n+2} \cdot 2^{n+1}} = \frac{\overset{+}{3^n} \cdot \overset{+}{2^n} \cdot \overset{+}{2}}{\underset{+}{2^{n+2}} \cdot \underset{+}{2^{n+1}}} > 0$$

$$a^m \cdot a^n = a^{m+n}$$

$$a^{m+n} = a^m \cdot a^n$$

8. Dano je zaporedje s splošnim členom $a_m = \frac{5m+1}{m^2+1}$. Pokaži, da je to zaporedje padajoče.

$$a_m = \frac{5m+1}{m^2+1}$$

$$a_{m+1} = \frac{5(m+1)+1}{(m+1)^2+1} = \frac{5m+5+1}{m^2+2m+1+1} = \frac{5m+6}{m^2+2m+2}$$

$$a_{m+1} - a_m = \frac{5m+6}{m^2+2m+2} - \frac{5m+1}{m^2+1} = \frac{(5m+6)(m^2+1)-(5m+1)(m^2+2m+2)}{(m^2+2m+2)(m^2+1)} =$$

$$= \frac{5m^3+5m+6m^2+6-(5m^3+10m^2+10m+m^2+2m+2)}{(m^2+2m+2)(m^2+1)} =$$

$$= \frac{\cancel{5m^3}+5m+6m^2+6-\cancel{5m^3}-10m^2-10m-m^2-2m-2}{(m^2+2m+2)(m^2+1)} =$$

$$= \frac{-5m^2-7m+4}{\underset{+}{(m^2+2m+2)}\underset{+}{(m^2+1)}}$$

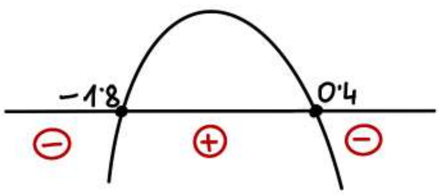
Ker je $m \in \mathbb{N}$ je $m^2+2m+2 > 0$ prav tako m^2+1

za kvadratno funkcijo v števcu pa moramo ugotoviti predznak

NIČLE: $-5m^2-7m+4=0$

$D = b^2-4ac = (-7)^2-4\cdot(-5)\cdot 4 = 129$

$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} \quad \begin{cases} x_1 \doteq -1{,}8 \\ x_2 \doteq 0{,}4 \end{cases}$$



za $x > 0{,}4$ je funkcija negativna
za $m \in \mathbb{N}$ je $-$

$$\frac{\overset{-}{-5m^2-7m+4}}{\underset{+}{(m^2+2m+2)}\underset{+}{(m^2+1)}} < 0$$

padajoče

9. Dano je zaporedje s splošnim členom $a_n = \frac{n-1}{n+1}$. Ali je dano zaporedje monotono?

$$a_n = \frac{n-1}{n+1}$$

$$a_{n+1} = \frac{n+1-1}{n+1+1} = \frac{n}{n+2}$$

$$a_{n+1} - a_n = \frac{n}{n+2} - \frac{n-1}{n+1} = \frac{n(n+1)-(n-1)(n+2)}{(n+2)(n+1)} =$$

$$= \frac{n^2+n-(n^2+2n-n-2)}{(n+2)(n+1)} = \frac{\cancel{n^2}+\cancel{n}-\cancel{n^2}-\cancel{2n}+\cancel{n}+2}{(n+2)(n+1)} =$$

$$= \frac{2^{\oplus}}{\underset{\oplus}{(n+2)}\underset{\oplus}{(n+1)}} > 0 \quad \text{za vsak } n \in \mathbb{N}$$

<u>naraščajoče</u> Zaporedje je monotono.

10. Dokaži, da je zaporedje s splošnim členom $a_n = \frac{2n+4}{3n+5}$ padajoče.

$$a_n = \frac{2n+4}{3n+5}$$

$$a_{n+1} = \frac{2(n+1)+4}{3(n+1)+5} = \frac{2n+2+4}{3n+3+5} = \frac{2n+6}{3n+8}$$

$$a_{n+1} - a_n = \frac{2n+6}{3n+8} - \frac{2n+4}{3n+5} = \frac{(2n+6)(3n+5)-(2n+4)(3n+8)}{(3n+8)(3n+5)} =$$

$$= \frac{6n^2+10n+18n+30-(6n^2+16n+12n+32)}{(3n+8)(3n+5)} =$$

$$= \frac{\cancel{6n^2}+\cancel{28n}+30-\cancel{6n^2}-\cancel{16n}-\cancel{12n}-32}{(3n+8)(3n+5)} = \frac{-2^{\ominus}}{\underset{\oplus}{(3n+8)}\underset{\oplus}{(3n+5)}} < 0 \quad \text{za vsak } n \in \mathbb{N}$$

<u>padajoče</u>

11. Ali je zaporedje s splošnim členom $a_n = \frac{n}{2n-7}$ monotono?

$a_n = \frac{n}{2n-7}$

$a_{n+1} = \frac{n+1}{2(n+1)-7} = \frac{n+1}{2n+2-7} = \frac{n+1}{2n-5}$

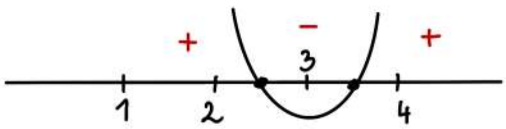
$a_{n+1} - a_n = \frac{n+1}{2n-5} - \frac{n}{2n-7} = \frac{(n+1)(2n-7) - n(2n-5)}{(2n-5)(2n-7)} =$

$= \frac{\cancel{2n^2} \cancel{-7n} + \cancel{2n} - 7 \cancel{-2n^2} + \cancel{5n}}{(2n-5)(2n-7)} = \frac{-7 \ominus}{(2n-5)(2n-7)}$

$2n-5$ je negativen za $n=1$ in $n=2$ ter pozitiven za $n \geq 3$
$2n-7$ je pozitiven za $n \geq 4$ $(n \in \mathbb{N})$

ugotovimo predznak kvadratne funkcije $(2n-5)(2n-7)$

NIČLE: $n_1 = \frac{5}{2} = 2{,}5$
$n_2 = 3{,}5$



za $n=1$ in $n=2$ je $+$
za $n=3$ je $-$
za $n \geq 4$ je $+$

Zaporedje ni monotono.

12. Pokaži, da je zaporedje $a_n = \frac{3^n}{n+3}$ naraščajoče.

$a_n = \frac{3^n}{n+3}$

$a_{n+1} = \frac{3^{n+1}}{n+1+3} = \frac{3^{n+1}}{n+4}$

Pokazati moraš, da velja: $a_{n+1} - a_n > 0$

$$a_{n+1} - a_n = \frac{3^{n+1}}{n+4} - \frac{3^n}{n+3} = \frac{3^{n+1}(n+3) - 3^n(n+4)}{(n+4)(n+3)} =$$

$$3^{n+1} = 3^n \cdot 3^1 = 3 \cdot 3^n$$

$$= \frac{3 \cdot 3^n(n+3) - 3^n \cdot n - 4 \cdot 3^n}{(n+4)(n+3)} = \frac{3 \cdot 3^n \cdot n + 9 \cdot 3^n - n \cdot 3^n - 4 \cdot 3^n}{(n+4)(n+3)} =$$

$$= \frac{3n \cdot 3^n - n \cdot 3^n + 5 \cdot 3^n}{(n+4)(n+3)} = \frac{2n \cdot 3^n + 5 \cdot 3^n}{(n+4)(n+3)} = \frac{3^n(2n+5)}{(n+4)(n+3)} > 0$$

$\oplus$ $\oplus$ $\oplus$ $\oplus$

za vsak $n \in \mathbb{N}$

NARAŠČAJOČE

13. Pokaži, da je zaporedje $a_n = \frac{2n+1}{3^n+1}$ padajoče.

$$a_n = \frac{2n+1}{3^n+1}$$

$$a_{n+1} = \frac{2(n+1)+1}{3^{n+1}+1} = \frac{2n+2+1}{3^{n+1}+1} = \frac{2n+3}{3^{n+1}+1}$$

$$3^{n+1} = 3^n \cdot 3$$

$$a_{n+1} - a_n = \frac{2n+3}{3^{n+1}+1} - \frac{2n+1}{3^n+1} = \frac{(2n+3)(3^n+1) - (2n+1)(3^{n+1}+1)}{(3^{n+1}+1)(3^n+1)} =$$

$$= \frac{2n \cdot 3^n + 2n + 3 \cdot 3^n + 3 - (2n+1)(3 \cdot 3^n + 1)}{(3^{n+1}+1)(3^n+1)} =$$

$$= \frac{2n \cdot 3^n + 2n + 3 \cdot 3^n + 3 - (6n \cdot 3^n + 2n + 3 \cdot 3^n + 1)}{(3^{n+1}+1)(3^n+1)} =$$

za vsak $n \in \mathbb{N}$
(1, 2, 3, 4 ...)

$$= \frac{2n \cdot 3^n + \cancel{2n} + \cancel{3 \cdot 3^n} + 3 - 6n \cdot 3^n - \cancel{2n} - \cancel{3 \cdot 3^n} - 1}{(3^{n+1}+1)(3^n+1)} = \frac{2 - 4n \cdot 3^n}{(3^{n+1}+1)(3^n+1)} < 0$$

$\ominus$ $\oplus$ $\oplus$

za $n \in \mathbb{N}$

PADAJOČE

14. Ali je zaporedje s splošnim členom $a_n = -2n^3 + n^2 + 2n + 1$ monotono?

$a_n = -2n^3 + n^2 + 2n + 1$

$a_{n+1} = -2(n+1)^3 + (n+1)^2 + 2(n+1) + 1 =$

$$\boxed{\begin{array}{l}(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \\ (a+b)^2 = a^2 + 2ab + b^2\end{array}}$$

$= -2(n^3 + 3n^2 + 3n + 1) + n^2 + \underline{2n} + \underline{1} + \underline{2n} + \underline{1} + \underline{1} =$

$= -2n^3 - \underline{6n^2} - \underline{6n} - \underline{2} + \underline{n^2} + \underline{4n} + \underline{3} = -2n^3 - 5n^2 - 2n + 1$

$a_{n+1} - a_n = -2n^3 - 5n^2 - 2n + 1 - (-2n^3 + n^2 + 2n + 1) =$

$= -\cancel{2n^3} - 5n^2 - 2n + \cancel{1} + \cancel{2n^3} - n^2 - 2n - \cancel{1} = -6n^2 - 2n = -2n(3n+1) < 0$

za vsak $n \in \mathbb{N}$

PADAJOČE

Zaporedje je monotono.

15. Pokaži, da je zaporedje $a_n = \log(3^n - 1)$ naraščajoče.

$a_n = \log(3^n - 1)$

$a_{n+1} = \log(3^{n+1} - 1)$

$$\log a - \log b = \log \frac{a}{b}$$

$a_{n+1} - a_n = \log(3^{n+1} - 1) - \log(3^n - 1) = \log \dfrac{3^{n+1} - 1}{3^n - 1} =$

$= \log \dfrac{3 \cdot 3^n - 1}{3^n - 1}$ — ulomek je večji od 1, ker je števec > imenovalec in sta oba pozitivna

$\log x > 0$, če je $x > 1$ in $\log x < 0$, če je $0 < x < 1$

$\log \dfrac{3 \cdot 3^n - 1}{3^n - 1} > 0$ za $n \in \mathbb{N}$ NARAŠČAJOČE